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A POSET ON MINIMAL COMPOSITIONS OF A POSITIVE INTEGER

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Abstract

In this paper, particular representatives of the orbits of the binary sequences of length n with respect to the action of the cyclic group C_n, in terms of ordered partitions of n, are determined. An algorithm that generates all these representatives is established and a poset on the set of these representatives is investigated. Moreover, a connection with subposets of the Boolean lattice is examined.

1. Ordered Partitions and (0, 1) Sequences

Let n be a positive integer. An ordered partition or composition of n is an ordered r-ple $\lambda = (k_1, k_2, \dots, k_r)$, where k_i are positive integers and $k_1 + k_2 + \dots + k_r = n$. The integers k_i are the parts of the partition.

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We associate with a composition $\lambda = (k_1, k_2, \ldots, k_r)$ a $(0, 1)$-sequence $s(\lambda)$ of length n, obtained from λ by replacing every part k_i with a sequence of $k_i - 1$ zeros and 1 one.

Conversely, given a $(0, 1)$-sequence $\mathbf{a} = a_1 a_2 \ldots a_n$, having $a_n = 1$, we associate with $\mathbf{a}$ an ordered partition of n, $p(\mathbf{a})$, where the first part is the number of zeros before the first one plus 1, the second part is the number of zeros between the first and the second one plus 1, and so on. Thus, $p(s(\lambda)) = \lambda$. The unique sequence, which does not correspond to a partition of n is that consisting of all zeros.

As example, let us consider the ordered partition of 7 $\lambda = (4, 1, 2)$. Then, $s(\lambda) = 0001101$. Conversely, if $\mathbf{a} = 0001101$, then $p(\mathbf{a}) = \lambda$.

We may assign a lexicographic order to the $(0, 1)$ n-sequences and ordered partitions in the following way.

We interpret the $(0, 1)$-sequences in base 2 and associate with every $(0, 1)$-sequence $\mathbf{a} = a_1 a_2 \ldots a_n$, the number

$$n(\mathbf{a}) = a_n + a_{n-1}.2 + \ldots + a_1.2^{n-1}.$$

A consequence is that the set of $(0, 1)$-sequences of length n is ordered by the linear order relation, which establishes

$$\mathbf{a} \leq \mathbf{b},$$

if and only if

$$n(\mathbf{a}) \leq n(\mathbf{b}),$$

where also $\mathbf{b}$ is a $(0, 1)$-sequence of length n.

A similar order relation holds for the set of ordered partitions of n.

Denoted $n(\lambda) = n(s(\lambda))$, we say

$$\lambda \leq \mu,$$

if and only if

$$n(\lambda) \leq n(\mu),$$

where λ and μ are ordered partitions of n.

A similar order relation for non ordered partitions is in [3].

2. Representatives of Orbits

Let $\mathbf{a} = a_1 a_2 \ldots a_n$ be a $(0, 1)$-sequence of length n. Given the permutation $\pi = (12 \ldots n)$, the action of π^i on $\mathbf{a}$ is

$$\pi^i(\mathbf{a}) = a_{1+i} a_{2+i} \ldots a_{i+n},$$

where $1 \leq i \leq n$, and the indices are modulo n. The orbits obtained under the action of the cyclic group C_n are called *binary necklaces*.

We choose as representative of each orbit, the element having least numerical value; in relation to the equivalence class determined by a sequence $\mathbf{a}$, we denote this value by $N(\mathbf{a})$.

In a similar way, given the composition $\lambda = (k_1, \ldots, k_r)$ and the permutation $\sigma = (12 \ldots r)$, the action of σ^j on λ is

$$\sigma^j(\lambda) = (k_{1+j}, \ldots, k_{r+j}),$$

where $1 \leq j \leq r$ and the indices are mod r.

We choose as representative of the orbit determined by the action of C_r, the value

$$N(\lambda) = N(s(\lambda)).$$

Let $P(n, r)$ be the set of orbits with respect to the action of C_r on the set of ordered partitions of n into r parts. The cardinality of $P(n, r)$, denoted by $p(n, r)$, turns out to be

$$p(n, r) = \frac{1}{n} . \sum_{d \mid r} \phi(d) . \binom{\frac{n}{d}}{\frac{r}{d}},$$

where ϕ is the Euler's function and $\begin{pmatrix} \frac{n}{d} \\ \frac{r}{d} \end{pmatrix}$ is defined to be zero, unless d is a divisor of both n and r [8]. Moreover, the total number of orbits is

$$p(n) = \sum_{r=1}^{n} p(n, r) = \frac{1}{n} . \sum_{d|n} \phi(d).2^{n|d} - 1.$$

Definition 1. A $(0, 1)$- sequence **a** is said **minimal**, if

$$n(\mathbf{a}) = N(\mathbf{a}).$$

Moreover, a composition λ is said **minimal**, if

$$n(\lambda) = N(\lambda).$$

In the following Theorem 1, we characterize minimal ordered partitions; as consequence by Proposition 1, we obtain a characterization of minimal $(0, 1)$ - sequences.

First, we establish some properties concerning the order relation between $(0, 1)$ - sequences and compositions.

Lemma 1. *Consider the* $(0, 1)$- *sequences* $\mathbf{a} = a_1 a_2 \dots a_n$ *and* $\mathbf{b} = b_1 b_2 \dots b_n$. *Then* $\mathbf{a} < \mathbf{b}$, *if and only if they coincide until a position q, after which* $a_{q+1} = 0$, *while* $b_{q+1} = 1$.

Proof. We only prove the sufficient condition. Assume that the condition is satisfied. Because $2^{q+1} > 1 + 2 + \dots + 2^q$, the result follows.□

Lemma 2. *If a* $(0, 1)$ - *sequence* $\mathbf{a} = a_1 a_2 \dots a_n$ *is minimal, then* $a_n = 1$.

Proof. If $a_n = 0$, then the sequence $\mathbf{b} = \pi^{-1}(\mathbf{a}) = a_n a_1 \dots a_{n-1}$ satisfies the contradictory relation $n(\mathbf{b}) < n(\mathbf{a})$. □

An immediate consequence of this lemma is the following result.

Proposition 1. *A* $(0, 1)$ - *sequence* **a** *is minimal, if and only if the composition* $p(\mathbf{a})$ *is minimal.*

Lemma 3. *Given two partitions* $\lambda = (k_1, \ldots, k_r)$ *and* $\mu = (t_1, \ldots, t_s)$ *of* n, $\lambda < \mu$, *if and only if either* $k_1 > t_1$ *or their parts coincide until a position d, after which* $k_{d+1} > t_{d+1}$.

Proof. The condition is equivalent to say that $s(\lambda) < s(\mu)$. ☐

Lemma 4. *Let* $\lambda = (k_1, \ldots, k_j, k_{j+1}, \ldots, k_r)$ *and* $\mu = (k_1, \ldots, k_j, t_{j+1}, \ldots, t_s)$ *be two partitions of* n, *whose first j parts coincide. Then* $\lambda < \mu$, *if and only if* $(k_{j+1}, \ldots, k_r) < (t_{j+1}, \ldots, t_s)$, *where* $(k_{j+1}, \ldots, k_r)$ *and* $(t_{j+1}, \ldots, t_s)$ *are partitions of* $n - (k_1 + k_2 + \cdots + k_j)$.

Proof. The numerical values of λ and μ depend on the different parts. ☐

From the above lemmas, we are able to obtain a characterization of the minimal ordered partitions of n, which turns out to be similar, but different from the notion of lexicographic composition given in [4].

Theorem 1. *An ordered partition* $\lambda = (k_1, k_2, \ldots, k_r)$ *of* n *is minimal*, *if and only if following properties hold*:

1. $k_1 \geq k_i$,

   $2 \leq i \leq n$;

2. *If, for suitable indices h and m,*

$$k_1 = k_{1+h},\ k_2 = k_{2+h},\ \ldots,\ k_m = k_{m+h},$$

and

$$k_{m+1} \neq k_{h+m+1},$$

then

$$k_{m+1} > k_{h+m+1},$$

where indices are mod r.

Proof. We only prove the necessary condition. Assume λ is minimal. If there exists a part k_j greater than k_1, then

$$\sigma^{j-1}(\lambda) = (k_j, k_{j+1}, \ldots, k_{j-1}),$$

where indices are mod r; by Lemma 3, this implies the impossible relation

$$n(\sigma^{j-1}(\lambda)) < n(\lambda).$$

In relation to the second property, assume that λ, and

$$\sigma^h(\lambda) = (k_{1+h}, k_{2+h}, \ldots, k_h),$$

coincide until a position k, after which $k_{m+1} < k_{m+1+h}$. By Lemma 3, it implies $\lambda > \sigma^h(\lambda)$, contradicting the assumption that λ is minimal. □

Definition 2. The part k_1 of a minimal composition $\lambda = (k_1, \ldots, k_r)$ is said the **main part** of λ.

Notice that the main part of a minimal composition can be repeated, as in the following example.

The ordered partition (4, 3, 2, 4, 3, 1) is minimal with main part 4, while the partition (4, 2, 3, 4, 3, 1) is not minimal.

The list of minimal partitions of $n = 4$ is: (4), (3, 1), (2, 2), (2, 1, 1), (1, 1, 1, 1).

3. Algorithm

Let L be the list of minimal ordered partitions of n. In this section, we present a constructive way for obtaining the list L, different from the algorithm examined in [4].

In particular, our aim is to determine the consecutive of a minimal partition μ of the list, which we denote μ^*.

We have to distinguish the cases, in which last part of $\mu = (a_1, a_2, \ldots, a_r)$ is greater than 1 or equal to 1.

1. Let $a_r > 1$. Then $\mu^* = (a_1, a_2, \ldots, a_r - 1, 1)$. By the assumption that μ is minimal, it follows that also the partition μ^* is minimal.

Indeed, it is immediate to prove that properties 1 and 2 of Theorem 1 are satisfied.

2. Let $a_r = 1$; in particular, $\mu = (a_1, a_2, \ldots, a_j, 1, \ldots, 1)$, with q final ones. We have to consider the following cases:

   (a) If $q + 1 < a_1$, then $\mu^* = (a_1, \ldots, (a_j - 1), (q + 1))$.

   (b) If $q + 1 = a_1$, then $\mu^* = (a_1, a_2, \ldots, (a_j - 1), (a_1 - 1), 1)$.

   (c) If $q + 1 > a_1$, then let us distinguish the cases:

   (i) $q + 1 < a_1 + \cdots + a_j - 1$.

   Let $a_1 + \cdots + a_s \leq q + 1 < a_1 + \cdots + a_{s+1}$.

   If $a_s > 1$, then, we have to consider the cases of $q + 1$ greater than or equal to $a_1 + a_2 + \cdots + a_s$.

   • Let $q + 1 > a_1 + a_2 + \cdots + a_s$. Then $\mu^* = (a_1, \ldots, (a_j - 1),$

   $a_1, \ldots, (a_s, t)$, where $t = q + 1 - (a_1 + \cdots + a_s)$. Notice that $t < a_{s+1}$.

   • Let $q + 1 = a_1 + a_2 + \cdots + a_s$. Then $\mu^* = (a_1, \ldots, (a_j - 1),$

   $a_1, \ldots, (a_s - 1), 1)$.

   Let $a_s = 1$; in particular, denote $a_1, \ldots, a_s = a_1, \ldots, a_k, 1, \ldots, 1$, where h is the number of last ones. Then, $\mu^* = (a_1, a_2, \ldots, (a_j - 1), a_1, \ldots, a_k - 1, c)$, where $c = q + 1 -(a_1 + \cdots + a_k - 1)$.

   (ii) $q + 1 \geq a_1 + \ldots + a_j - 1$.

   Denote d, the quotient of the division $\dfrac{q+1}{a_1 + \ldots + a_j - 1}$ and d' the rest.

If $d' < a_1$, then denoted $\alpha = a_1 \dots a_j - 1 \cdots a_1 \dots a_j - 1$, where the sequence $a_1 \dots a_j - 1$ is repeated d times, we have $\mu^* = \alpha d'$.

If $d' = a_1$, then $\mu^* = (\alpha, d' - 1, 1)$.

If $d' > a_1$, and in particular, $a_1 + \cdots + a_s \le d' < a_1 + \cdots + a_{s+1}$, then we may repeat the first case in relation to d'.

It is possible to see that $n(\mu^*) > n(\mu)$; moreover, if k is an odd integer between $n(\mu)$ and $n(\mu^*)$ and $k = n(\alpha)$, then α is not minimal.

For example, the list L for $n = 6$ is (6), (5, 1), (4, 2), (4, 1, 1), (3, 3), (3, 2, 1), (3, 1, 2), (3, 1, 1, 1), (2, 2, 2), (2, 2, 1), (2, 1, 2, 1), (2, 1, 1, 1, 1), (1, 1, 1, 1, 1, 1).

These compositions correspond, respectively, to the numbers 1, 3, 5, 7, 9, 11, 13, 15, 21, 23, 27, 31, 63. Notice that, the missing number 17 corresponds to the sequence (0, 1, 0, 0, 0, 1) associated to the composition (2, 4), which is not minimal.

It is easy to see that the other missing values correspond to compositions, which are not minimal.

4. Poset

In this section, we consider a partial order defined on the set P_n of all minimal ordered partitions of a positive integer n.

We say that a composition β covers a composition α, if β can be obtained by replacing two cyclically consecutive parts of α by their sum.

For example, for $n = 7$, the ordered partition $\alpha = (3, 1, 2, 1)$ is covered by $\beta = (4, 1, 2)$, because the part 4 is obtained by summing last part 1 with the first part 3.

We define the partial order $(P_n, \ll)$ as the reflexive and transitive closure of the "is covered by" relation. The partial order is said a cyclic refinement and the corresponding poset P_n is said the *poset of minimal compositions*.

A collection of partitions, no two of which are related by refinement forms an antichain. If β covers α, then the latter contains strictly more parts than the former. Thus, for each k, the set $P(n, k)$ of all partitions having k parts is an antichain. Moreover, these sets determine the levels, into which the poset may be partitioned. It implies that the poset is ranked and the rank of P_n is the number n of levels.

The height is $n - 1$, because starting from the partition $1, 1, \ldots, 1$, it needs $n - 1$ sums to arrive to the partition n.

In [6], it is proved that $p(n, i) = p(n, n - i)$; then the following result holds, where $p_i = p(n, i)$.

Proposition 2. *Let* $n = 2t + 1$ *odd. The sequence* $p_2, p_3, \ldots, p_n$ *is symmetric.*

Recall [1] that a bipartite poset is an ordered triple $(X, Y; \leq)$, where X and Y are disjoint and $x \leq y$ implies that $x \in X$ and $y \in Y$.

X and Y also form an antichain with respect to the order relation $\leq$. The poset $(P_n, \ll)$ is bipartite; indeed, the set of minimal partitions is partitioned into the subsets X and Y, which contain the partitions of n into an even or odd number of parts, respectively. Clearly, when a partition α covers a partition β, their numbers of parts have different parity.

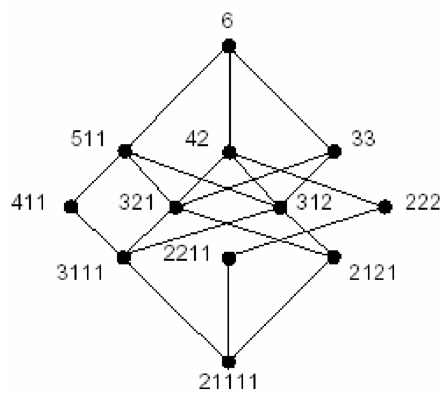

Figure 1. The Hasse diagram of P_6, without the composition $(1, \ldots, 1)$, covered by (2, 1, 1, 1, 1).

5. Subposet of the Boolean Lattice

Recall that the Boolean lattice B_n, is the partially ordered set of subsets of $[n] = \{1, 2, \ldots, n\}$ ordered by inclusion. The Hasse diagram of B_n is isomorphic to the n-cube, whose vertices are the binary n-strings, with two strings adjacent, if and only if they differ in one position.

The i-th level of B_n can thus be viewed as the set of n-bit binary strings with i ones. Let Q_n be the set of representatives of the orbits under the action of π on the set of all the binary n-strings.

We say that one composition β covers a composition α, if the former can be obtained by replacing two consecutive parts of α by their sum. The difference with respect to the previous poset $(P_n, \ll)$ is that now, we have deleted the "cyclically" condition.

We define the partial order $(Q_n, \preceq)$ as the reflexive and transitive closure of the preceding "is covered by" relation.

If we replace every composition $\lambda \in Q_n$ by the string $s(\lambda)$, we obtain a set Q'_n, in which we may define a partial order $\preceq$ such that $s(\alpha) \leq s(\beta)$, when $\alpha \preceq \beta$ and in correspondence the poset $(Q'_n, \preceq)$.

Theorem 2. $(Q'_n, \preceq)$ *is a subposet of* B_n.

Proof. Let $\alpha = (a_1, a_2, \ldots, a_r)$ and $\beta = (b_1, b_2, \ldots, b_s)$ be two compositions of n, satisfying the condition that β covers α, with respect to the order relation $\preceq$. This implies that the sequences of the parts of these compositions coincide, but one part of β, which is the sum of two consecutive parts of α.

Our aim is to prove that the strings $a = s(\alpha)$ and $b = s(\beta)$ have Hamming distance 1.

Without loss of generality, we may assume that $b_1 = a_1 + a_2$, $b_2 = a_3, \ldots, b_s = a_r$. Then, the first $a_1 + a_2$ elements of b coincide with the corresponding elements of a, but for the element in position a_1, which holds 1 in a and 0 in b. Because all the remaining elements of the strings coincide, then the Hamming distance holds 1, and these strings are adjacent in the Hasse diagram of B_n. It implies that Q'_n turns out to be a subposet of B_n. □

Corollary 1. $(P_n, \ll)$ *is rank unimodal.*

Proof. It is an immediate consequence of the fact that the levels of P_n and Q_n coincide and the fact that Q'_n, a subposet of the Boolean lattice, is rank unimodal. □

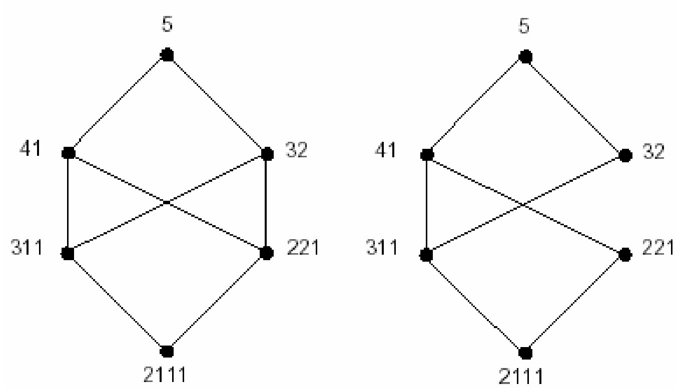

Figure 2. The Hasse diagrams of $(P_5, \ll)$ and $(Q_5, \preceq)$, without the partition $(1, \ldots, 1)$, covered by (2, 1, 1, 1).

In this case, the elements of Q_5' are the strings $s(5) = 00001$, $s(4, 1) = 00011$, $s(3, 2) = 00101$, $s(3, 1, 1) = 00111$, $s(2, 2, 1) = 01011$, $s(2, 1, 1, 1) = 01111$.

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